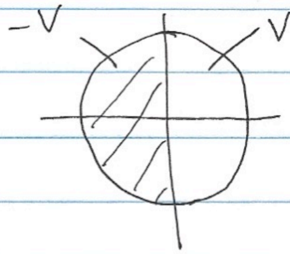
Jackson 3.10 (a)

We have $\Phi(\rho,\phi,z) = \sum_{n,m} [A_{nm} \sin m\phi + B_{nm} \cos m\phi] \sin(\frac{n\pi}{L} z) I_m(\frac{n\pi}{L}\rho)$

~~Orthogonality~~ For $V(\phi,z) = \begin{cases} V & -\pi/2 < \phi < \pi/2 \\ -V & \pi/2 < \phi < 3\pi/2 . \end{cases}$

we can see that $A_{nm} = 0$ because it's antisymmetric about $\phi = 0$,

so we can simplify the expression:



$$\Phi = \sum_{n,m} \left(B_{nm} \cos m\phi\right) \sin(\frac{n\pi}{L} z) I_m(\frac{n\pi}{L}\rho)$$

Orthogonality gives $\int_0^{2\pi} d\phi \, \Phi(\rho,\phi,z) \cos m\phi = \pi \overbrace{B_{nm}}^{\sum_n} \sin(\frac{n\pi}{L} z) I_m(\frac{n\pi}{L}\rho)$

For the given potential, $\int_0^{2\pi} d\phi \, \Phi(\rho,\phi,z)\Big|_{\rho=b} \cos m\phi$

$$= \int_{-\pi/2}^{\pi/2} V \cos m\phi \, d\phi + \int_{\pi/2}^{3\pi/2} (-V) \cos m\phi \, d\phi$$

$$= \frac{4V}{m} \sin(\frac{m\pi}{2})$$

$$= \pi \cancel{B_{nm}} \sum_n B_{nm} \sin(\frac{n\pi}{L} z) I_m(\frac{n\pi}{L} b)$$

Organizing terms: $\sum_n B_{nm} \sin(\frac{n\pi}{L} z) I_m(\frac{n\pi}{L} b) = \frac{4V}{m\pi} \sin(\frac{m\pi}{2})$

Applying $\int_0^L \sin^2\left(\frac{n\pi}{L}z\right)dz = \frac{L}{2}$, we have

$$\int_0^L \frac{4V}{m\pi}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{L}z\right)dz = \frac{L}{2}B_{nm}I_m\left(\frac{n\pi}{L}b\right)$$

$$B_{nm} = \frac{8V}{m\pi L}\frac{\sin\left(\frac{m\pi}{2}\right)}{I_m\left(\frac{n\pi}{L}b\right)}\left[\int_0^L \sin\left(\frac{n\pi}{L}z\right)dz\right]$$

$$\Downarrow$$

$$= \frac{8V}{m\pi \cancel{L}}\frac{\sin\left(\frac{m\pi}{2}\right)}{I_m\left(\frac{n\pi}{L}b\right)}\left[\frac{\cancel{L}}{n\pi}\left(1-\cos(n\pi)\right)\right]$$

$$= \frac{8V}{\pi^2 mn}\frac{1}{I_m\left(\frac{n\pi}{L}b\right)}\sin\left(\frac{m\pi}{2}\right)\left(1-\cos n\pi\right)$$

Observing $\sin\left(\frac{m\pi}{2}\right)\left(1-\cos n\pi\right) = \begin{cases} 2(-1)^{(m-1)/2} & \text{if } n,m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow B_{nm} = \begin{cases} \frac{16V}{\pi^2 mn}\frac{1}{I_m\left(\frac{n\pi}{L}b\right)}(-1)^{(m-1)/2} & \text{if } n,m \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

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